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Static axially symmetric gravitational fields with shell sources

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Abstract. Israel's method for treating surface layers in general relativity is applied to construct shell sources for exterior static axially symmetric gravitational fields. Consideration is restricted to cases in which the 3-cylinder representing the history of the shell is an equipotential surface of the exterior field and consequently the space–time inside this 3-cylinder is flat.

1. Introduction

An important problem in general relativity is that of constructing physically realistic sources for known vacuum solutions of Einstein's field equations. In the present work, the method of Israel (1966) is used to construct shell sources for the Weyl class of static axially symmetric fields (see Synge 1964, p 312). More precisely, space–times are exhibited for which the metrics inside and outside a certain 3-cylinder (the history of a closed shell) are of the Weyl class and satisfy the appropriate junction conditions across the 3-cylinder; furthermore the resulting energy tensor of the shell is physically realistic.

In their study of similar problems, Morgan and Morgan (1970) renounce Israel's method in favour of a more direct use of the field equations which is made possible by the high degree of symmetry involved. In doing so they claim that 'it is possible to gain a greater insight into the problem'. While acknowledging that their method does illuminate many aspects of the problem, it is the hope of the present author to show that Israel's method provides equally valuable insights which complement those of Morgan and Morgan.

It may validly be argued, of course, that shell sources are not very realistic, particularly if one is interested in astrophysical applications. However, it is not unreasonable to think that their properties should provide at least some qualitative indications as to how a more realistic source might behave.

In § 2 an outline of Israel's method is given and this is applied to the case of axially symmetric fields in § 3. By restricting our attention to cases in which the shell lies on an equipotential surface of the exterior field, in a sense to be defined, it is found that the problem is considerably simplified. In §§ 4 and 5, having chosen a particular exterior solution of Weyl's equations in terms of prolate spheroidal coordinates, the various possible 'equipotential' shell sources for this field are examined. Finally in § 6, the Curzon metric and the monopole solution of Zipoy (1966) in oblate spheroidal coordinates are briefly examined.

2. Thin shells

Let the 3-cylinder Σ be the history of a closed thin shell in space–time, V^- and V^+ the vacuum space–times inside and outside the shell respectively. Let

$$\{x_+^i\}, \{x_-^i\} \qquad (i=0,1,2,3)$$

be coordinates in V^+ (respectively V^-) in a neighbourhood of Σ and let ξ^μ ($\mu=0,1,2$) be intrinsic coordinates on Σ. The equation of Σ regarded as embedded in V^+ will be of the form $x_+^i = x_+^i\,(\xi^\mu)$ and its equation in V^- will be $x_-^i = x_-^i\,(\xi^\mu)$. The metric of V^+ is $ds_+^2 = g_{ij}^+(x_+)\,dx_+^i\,dx_+^j$ and that of V^- is $ds_-^2 = g_{ij}^-(x_-)\,dx_-^i\,dx_-^j$.

The first fundamental forms (or intrinsic metrics) on Σ, induced by its embedding in V^+ and V^- respectively, are

$$(ds_+^2)_\Sigma = g_{ij}^+\frac{\partial x_+^i}{\partial\xi^\mu}\frac{\partial x_+^j}{\partial\xi^\nu}\,d\xi^\mu\,d\xi^\nu$$

and

$$(ds_-^2)_\Sigma = g_{ij}^-\frac{\partial x_-^i}{\partial\xi^\mu}\frac{\partial x_-^j}{\partial\xi^\nu}\,d\xi^\mu\,d\xi^\nu \tag{2.1}$$

where g_{ij}^+ and g_{ij}^- are evaluated on Σ. The second fundamental forms (or extrinsic curvatures) of Σ in V^+, V^- respectively are given by

$$K_{\mu\nu}^+ = n_{i|j}^+\frac{\partial x_+^i}{\partial\xi^\mu}\frac{\partial x_+^j}{\partial\xi^\nu}, \qquad K_{\mu\nu}^- = n_{i|j}^-\frac{\partial x_-^i}{\partial\xi^\mu}\frac{\partial x_-^j}{\partial\xi^\nu}, \tag{2.2}$$

where n_i^+ (respectively n_i^-) are the covariant components of the unit vector normal to Σ in V^+ (respectively V^- and pointing out from (respectively into) Σ.

Since Σ is the history of a shell we have, following Israel (1966),

$$(ds_+^2)_\Sigma = (ds_-^2)_\Sigma. \tag{2.3}$$

Furthermore, defining $\gamma_{\mu\nu}$ by

$$\gamma_{\mu\nu} = K_{\mu\nu}^+ - K_{\mu\nu}^- \tag{2.4}$$

the *surface energy tensor* $S_{\mu\nu}$ of the shell is given by the 'Lanczos equations',

$$-\kappa S_{\mu\nu} = \gamma_{\mu\nu} - g_{\mu\nu}\gamma, \tag{2.5}$$

$g_{\mu\nu}$ being the intrinsic metric tensor on Σ, $\gamma = \gamma_\mu^\mu$ and $\kappa = 8\pi$. The relation of $S_{\mu\nu}$ to the (singular) energy tensor of the matter constituting the shell is clarified in the appendix.

3. Axially symmetric fields

Consider the case in which V^+ and V^- are static, vacuum, axially symmetric space-times. By choosing Weyl's quasi-cylindrical coordinates (r, z, ϕ, t) in V^+ and $(\bar r, \bar z, \bar\phi, \bar t)$

in V^-, the line elements may be written in the form (see Synge 1964, p 312)

$$ds_+^2 = e^{2(\nu-\lambda)}(dr^2+dz^2)+r^2 e^{-2\lambda} d\phi^2 - e^{2\lambda} dt^2 \tag{3.1}$$

$$ds_-^2 = e^{2(\bar\nu-\bar\lambda)}(d\bar r^2+d\bar z^2)+\bar r^2 e^{-2\bar\lambda} d\bar\phi^2 - e^{2\bar\lambda} d\bar t^2 \tag{3.2}$$

respectively, where $\lambda = \lambda(r, z)$, $\nu = \nu(r, z)$,

$$\frac{\partial^2\lambda}{\partial r^2}+\frac{\partial^2\lambda}{\partial z^2}+\frac{1}{r}\frac{\partial\lambda}{\partial r}=0, \tag{3.3}$$

$$\frac{\partial\nu}{\partial r}=r\left[\left(\frac{\partial\lambda}{\partial r}\right)^2-\left(\frac{\partial\lambda}{\partial z}\right)^2\right], \qquad \frac{\partial\nu}{\partial z}=2r\frac{\partial\lambda}{\partial r}\frac{\partial\lambda}{\partial z}, \tag{3.4}$$

with identical equations for $\bar\lambda(\bar r, \bar z)$ and $\bar\nu(\bar r, \bar z)$.

Let $z=f(r)$ be the equation of Σ in V^+ and $\bar z = \bar f(\bar r)$ its equation in V^-. The intrinsic metrics of Σ in V^+ and V^- are then

$$(ds_+^2)_\Sigma = e^{2(\nu-\lambda)}(1+f'^2)\, dr^2+r^2 e^{-2\lambda} d\phi^2 - e^{2\lambda} dt^2 \tag{3.5}$$

and

$$(ds_-^2)_\Sigma = e^{2(\bar\nu-\bar\lambda)}(1+\bar f'^2)\, d\bar r^2+\bar r^2 e^{-2\bar\lambda} d\bar\phi^2 - e^{2\bar\lambda} d\bar t^2, \tag{3.6}$$

respectively, where a prime denotes derivative with respect to r (respectively $\bar r$). By (2.3), these two expressions must agree, i.e. there must be a transformation from (r, ϕ, t) to $(\bar r, \bar\phi, \bar t)$ under which (3.5) transforms to (3.6). Since the metric components are independent of ϕ, t and $\bar\phi$, $\bar t$ and there are no cross terms, it is clear that by a trivial adjustment we can make $\bar\phi = \phi$ and $\bar t = t$. Hence, comparing (3.5) and (3.6) we obtain, on Σ,

$$\bar r = r, \qquad \bar\lambda = \lambda, \qquad e^{2\bar\nu}(1+\bar f'^2) = e^{2\nu}(1+f'^2). \tag{3.7}$$

Given λ and ν in V^+ and the equation $z=f(r)$ of Σ in V^+ we can, in principle, determine $\bar\lambda$ and $\bar\nu$ in V^- and the equation $\bar z = \bar f(r)$ of Σ in V^- by using the simultaneous system of equations (3.3), (3.4) (for $\bar\lambda$, $\bar\nu$) and the second and third equations of (3.7). However, in practice, it is difficult to disentangle this system of equations for the general case.

Things become somewhat simpler if we take the shell to be an 'equipotential' surface of the external field in the sense that

$$(\lambda)_\Sigma = \lambda_0 \text{ (constant)}. \tag{3.8}$$

By the second of (3.7) we then have $(\bar\lambda)_\Sigma = \lambda_0$ and hence, since $\bar\lambda$ is a harmonic function, $\bar\lambda = \lambda_0$ in V^- (see Courant and Hilbert 1962, p 255). By equation (3.4) (for $\bar\lambda$, $\bar\nu$) $\bar\nu$ is also constant in V^- and, since elementary flatness requires that $\bar\nu = 0$ on the axis of symmetry (Synge 1964, p 314), it follows that $\bar\nu = 0$ everywhere in V^-. The interior metric is therefore

$$ds_-^2 = e^{-2\lambda_0}(d\bar r^2+d\bar z^2+\bar r^2\, d\phi^2) - e^{2\lambda_0}\, dt^2 \tag{3.9}$$

which is Minkowskian. Equation (3.7) then becomes

$$(1+\bar f'^2) = e^{2\nu}(1+f'^2), \qquad \text{on } \Sigma, \tag{3.10}$$

and, since ν and f are given, we can determine the equation $\bar z = \bar f(r)$ of Σ in V^-. By using the equations (2.2) it is now a straightforward matter to calculate the second fundamental forms $K^{+}_{\mu\nu}$ and $K^{-}_{\mu\nu}$, where $(\xi^1, \xi^2, \xi^3) = (r, \phi, t)$. Equations (2.4) and (2.5) then yield the energy tensor $S_{\mu\nu}$ of the shell.

An interesting feature of (3.10) is that it admits a solution for $\bar{f}$ only if

$$e^{2\nu}(1+f'^2)-1 \geqslant 0, \qquad \text{on } \Sigma. \tag{3.11}$$

Hence, if the inequality (3.11) does not hold, Σ cannot be embedded in a flat interior V^-. We shall see that the effect of this is to impose an upper limit on the ratio of the mass to a typical radius of the shell.

In what follows, we shall refer to a shell lying on a hypersurface Σ which satisfies property (3.8) as an 'equipotential shell source'.

4. Prolate spheroidal shells

In this section the method of § 3 is applied to a particular solution of Weyl's equations in V^+. The interest of the solution which we consider, as distinct, for instance, from those mentioned in § 6, lies in the wider range of possible equipotential shell sources which it allows.

We firstly transform from Weyl's quasi-cylindrical coordinates (r, z, ϕ, t) to quasi-prolate spheroidal coordinates (x, y, ϕ, t) defined by

$$r = a(x^2-1)^{1/2}(1-y^2)^{1/2}, \qquad z = axy, \qquad \phi = \phi, \qquad t = t, \tag{4.1}$$

where a is a constant and the ranges of the coordinates x, y are $1 \leqslant x < \infty$, $-1 \leqslant y \leqslant 1$. The surfaces $x =$ constant are prolate spheroids with major axes along the z axis while the surfaces $y =$ constant are two-sheeted hyperboloids of revolution with the z axis as axis of symmetry.

The Laplace equation (3.3) for λ is then

$$\frac{\partial}{\partial x}\left((x^2-1)\frac{\partial \lambda}{\partial x}\right)+\frac{\partial}{\partial y}\left((1-y^2)\frac{\partial \lambda}{\partial y}\right)=0, \tag{4.2}$$

while the equations for ν are

$$\begin{aligned}\frac{\partial \nu}{\partial x}&=\left(\frac{1-y^2}{x^2-y^2}\right)\left[x(x^2-1)\left(\frac{\partial \lambda}{\partial x}\right)^2-x(1-y^2)\left(\frac{\partial \lambda}{\partial y}\right)^2-2y(x^2-1)\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial y}\right]\\ \frac{\partial \nu}{\partial y}&=\left(\frac{x^2-1}{x^2-y^2}\right)\left[y(x^2-1)\left(\frac{\partial \lambda}{\partial x}\right)^2-y(1-y^2)\left(\frac{\partial \lambda}{\partial y}\right)^2+2x(1-y^2)\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial y}\right].\end{aligned} \tag{4.3}$$

The solutions of (4.2) which are regular on the semi-infinite line-segments $y = \pm 1$ and well-behaved at infinity are of the form

$$\lambda = \sum_l A_l Q_l(x) P_l(y) \tag{4.4}$$

where $P_l(y)$, $l = 0, 1, \ldots,$ are the Legendre polynomials and $Q_l(x)$ the Legendre functions of the second kind.

Let us take the solution of (4.2) with $l = 0$, i.e.

$$\lambda = -\tfrac{1}{2}\beta \log\frac{x+1}{x-1}, \tag{4.5}$$

where β is a positive constant (the factor $\frac{1}{2}$ is there for later convenience). Substituting this in (4.3) and solving, we obtain

$$\nu=\tfrac{1}{2}\beta^2\log\frac{x^2-1}{x^2-y^2} \tag{4.6}$$

where the constant of integration has been chosen so that $\nu=0$ on the axis of symmetry ($y=\pm 1$). The metric (3.1) in V^+ is therefore

$$\begin{aligned}\mathrm{d}s_+^2=a^2\left(\frac{x+1}{x-1}\right)^{\beta}\left(\frac{x^2-1}{x^2-y^2}\right)^{\beta^2}(x^2-y^2)\left(\frac{\mathrm{d}x^2}{x^2-1}+\frac{\mathrm{d}y^2}{1-y^2}\right)\\ +a^2\left(\frac{x+1}{x-1}\right)^{\beta}(x^2-1)(1-y^2)\,\mathrm{d}\phi^2-\left(\frac{x-1}{x+1}\right)^{\beta}\mathrm{d}t^2.\end{aligned} \tag{4.7}$$

Let the history of the shell, Σ, be the 'prolate spheroidal' 3-cylinder $x=x_0$ (constant) >1. Then the intrinsic metric of Σ due to its embedding in V^+ is

$$\begin{aligned}(\mathrm{d}s_+^2)_\Sigma=a^2\left(\frac{x_0+1}{x_0-1}\right)^{\beta}\left(\frac{x_0^2-1}{x_0^2-y^2}\right)^{\beta^2}(x_0^2-y^2)\frac{\mathrm{d}y^2}{1-y^2}+a^2\left(\frac{x_0+1}{x_0-1}\right)^{\beta}(x_0^2-1)(1-y^2)\,\mathrm{d}\phi^2\\ -\left(\frac{x_0-1}{x_0+1}\right)^{\beta}\mathrm{d}t^2,\end{aligned} \tag{4.8}$$

where y, ϕ and t are taken as intrinsic coordinates on Σ. Since Σ is an equipotential surface, in the sense already defined, the line element in V^- is that of (3.9) which in the present case becomes

$$\mathrm{d}s_-^2=\left(\frac{x_0+1}{x_0-1}\right)^{\beta}(\mathrm{d}\bar{r}^2+\mathrm{d}\bar{z}^2+\bar{r}^2\,\mathrm{d}\phi^2)-\left(\frac{x_0-1}{x_0+1}\right)^{\beta}\mathrm{d}t^2. \tag{4.9}$$

In terms of the intrinsic coordinates (y,ϕ,t) on Σ, the equation of the latter in V^- will be of the form

$$\bar{r}=f(y),\qquad \bar{z}=g(y), \tag{4.10}$$

where $f(y)$ and $g(y)$ are to be determined by the junction conditions (2.3). The intrinsic metric of Σ induced by its embedding in V^- is then

$$(\mathrm{d}s_-^2)_\Sigma=\left(\frac{x_0+1}{x_0-1}\right)^{\beta}(f'^2+g'^2)\,\mathrm{d}y^2+f^2\left(\frac{x_0+1}{x_0-1}\right)^{\beta}\mathrm{d}\phi^2-\left(\frac{x_0-1}{x_0+1}\right)^{\beta}\mathrm{d}t^2, \tag{4.11}$$

and since, by (2.3), this must agree with (4.8), we obtain

$$f(y)=a(x_0^2-1)^{1/2}(1-y^2)^{1/2} \tag{4.12}$$

in agreement with the already known fact that $\bar{r}=r$ on Σ, and

$$g'^2(y)=\frac{a^2(x_0^2-1)}{1-y^2}\left[\left(\frac{x_0^2-1}{x_0^2-y^2}\right)^{\beta^2-1}-y^2\right]. \tag{4.13}$$

In the subsequent work we shall not need an explicit expression for $g(y)$, so we can sidestep the daunting task of integrating (4.13). However, in order that a real function $g(y)$ should exist, it is clear that we must have

$$\frac{a^2(x_0^2-1)}{1-y^2}\left[\left(\frac{x_0^2-1}{x_0^2-y^2}\right)^{\beta^2-1}-y^2\right]\geqslant 0. \tag{4.14}$$

To examine the conditions under which the inequality (4.14) holds, we distinguish the three possibilities, (*a*) $\beta=1$, (*b*) $\beta>1$ and (*c*) $\beta<1$.

In case (*a*) ($\beta=1$), Erez and Rosen (1959) have pointed out that the metric in V^+ is that of Schwarzschild. To see this put $\beta=1$ in (4.7), let $a=m$ and make the transformation $x=(r/m)-1$, $y=\cos\theta$ to get

$$ds_+^2=\left(1-\frac{2m}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)-\left(1-\frac{2m}{r}\right)dt^2.$$

By (4.12) and (4.13) (with $\beta=1$), the equation of Σ in V^- becomes

$$\bar{r}=m(x_0^2-1)^{1/2}(1-y^2)^{1/2}, \qquad \bar{z}=m(x_0^2-1)^{1/2}y \tag{4.15}$$

where we take $\bar{z}=0$ at $y=0$. In other words Σ is the sphere

$$\bar{r}^2+\bar{z}^2=m^2(x_0^2-1). \tag{4.16}$$

If we make the transformation $R=[(x_0+1)/(x_0-1)]^{1/2}\bar{r}$, $Z=[(x_0+1)/(x_0-1)]^{1/2}\bar{z}$, $T=[(x_0-1)/(x_0+1)]^{1/2}t$ in (4.9) (with $\beta=1$), we obtain, in V^-, the more familiar flat space–time metric

$$ds_-^2=dR^2+dZ^2+R^2\,d\phi^2-dT^2 \tag{4.17}$$

and the equation (4.15) for Σ in V^- becomes

$$R^2+Z^2=m^2(x_0+1)^2. \tag{4.18}$$

Thus the radius D_0 of the sphere Σ in the interior Euclidean space is

$$D_0=m(x_0+1) \tag{4.19}$$

and as $x_0\to 1$, $D_0\to 2m$, the Schwarzschild radius. We shall see subsequently that the stresses in the shell become infinite in this limit.

In case (*b*) ($\beta>1$), the inequality (4.14) holds, for all y in $-1\leqslant y\leqslant 1$, only if $x_0\geqslant\beta$. It will appear later that the mass of the shell is $M=\beta a$, so that this inequality becomes $(M/ax_0)\leqslant 1$. Thus the restriction on the possibility of embedding Σ in V^- expressed by (4.14) puts an upper bound on the ratio of the mass to a typical radius of the shell.

In case (*c*) ($\beta<1$), the inequality (4.14) holds for all y in $-1\leqslant y\leqslant 1$ and for all $x_0\geqslant 1$. Thus we can have a complete sequence of equipotential shell sources, $x=x_0$, with $1\leqslant x_0<\infty$. Examination of (4.11)–(4.13) then shows that, as x_0 approaches unity, the proper radius of the shell in the r direction goes to zero whereas it becomes infinitely long in the z direction. Furthermore, the proper area of the shell tends to zero. Thus in the limit, as x_0 approaches unity, the shell becomes a thin rod of infinite length.

So far, we have examined the possible equipotential shell sources, for the exterior metric (4.7), which are allowed geometrically. Our next task is to inquire into the restrictions, if any, imposed by the requirement that the energy tensor of the shell should be physically realistic.

5. The surface energy tensor of the shell

A straightforward calculation, using equations (2.2)–(2.4) with intrinsic coordinates $(\xi^0, \xi^2, \xi^3)=(t, y, \phi)$ on Σ, yields the following mixed components of the surface energy tensor of the shell:

$$\kappa S_0^0 = A(y)[B(y)+C(y)-2\beta(x_0^2-y^2)], \tag{5.1}$$

$$\kappa S_2^2 = A(y)C(y), \tag{5.2}$$

$$\kappa S_3^3 = A(y)B(y), \tag{5.3}$$

where

$$\begin{aligned} A(y) &= [a(x_0^2-1)(x_0^2-y^2)]^{-1}\left(\frac{x_0-1}{x_0+1}\right)^{\frac{1}{2}\beta}\left(\frac{x_0^2-y^2}{x_0^2-1}\right)^{\frac{1}{2}(\beta^2-1)}, \\ B(y) &= \beta^2 x_0(1-y^2)+x_0(x_0^2-1)-[H(y)]^{-1}(x_0^2-\beta^2y^2)(x_0^2-1), \\ C(y) &= (x_0^2-y^2)[x_0-H(y)] \end{aligned} \tag{5.4}$$

and

$$H(y) = \left(\frac{x_0^2-1}{1-y^2}\right)^{1/2}\left[\left(\frac{x_0^2-1}{x_0^2-y^2}\right)^{(\beta^2-1)} - y^2\right]^{1/2}. \tag{5.5}$$

Adapting Whittaker's (1935) theorem to the case of a surface layer (see appendix), we may immediately calculate the total mass M of the shell by the integral,

$$M = \int_\Sigma (-S_0^0+S_2^2+S_3^3)\sqrt{-g^{(3)}}\,\mathrm{d}y\,\mathrm{d}\phi, \tag{5.6}$$

where $g^{(3)}$ is the determinant of the intrinsic metric on Σ. This yields

$$M = \beta a \tag{5.7}$$

which is identical with the result obtained by Bonnor and Sackfield (1968) for a disc.

In order that (5.1)–(5.3) should represent the components of a physically realistic energy tensor for the cases (a), (b) and (c) of § 4 we require that, for the full ranges of the parameters x_0 and β and of the variable y in each of these cases,

(i) $S_0^0 \leqslant 0$ (the weak energy condition, see Hawking and Ellis 1973, pp 89–95),

(ii) $-S_0^0+S_2^2+S_3^3 \geqslant 0$ (the strong energy condition, again see Hawking and Ellis reference),

(iii) for a weak field, i.e. $(\beta/x_0) \ll 1$, $(|S_2^2|/|S_0^0|)$ and $(|S_3^3|/|S_0^0|)$ should be of order (β/x_0).

It may be verified that these conditions are, in fact, satisfied. Furthermore, S_2^2 and S_3^3 are always positive so that they represent pressures.†

It is of interest to examine, in each of the above cases, what happens to the energy tensor as the limiting surface for that case is approached. In case (a) $(\beta = 1)$ where, as x_0 tends to unity, the radius of the sphere Σ approaches the Schwarzschild radius, the pressures become infinite and the density, $-S_0^0$, tends to $(1/8\pi m)$. In case (b) $(\beta > 1)$, as x_0 approaches β, the density and pressures remain finite, so nothing catastrophic occurs. For values of x_0 less than β one gets imaginary values for the energy tensor components. In case (c) $(\beta < 1)$ where, as x_0 tends to unity, the prolate spheroid becomes an infinitely long thin rod lying along the z axis, both the density and the pressures tend to infinity.

† Conditions (ii), (iii) and the positive character of S_2^2 are easily verified. For condition (i) and the positive character of S_3^3, the author must admit to having had recourse to a computer.

6. Some other models

The method described in § 3 is clearly applicable to a wide variety of vacuum axisymmetric metrics of V^+ of which only one has been considered. Some results for two further cases are sketched in the present section.

If, instead of (4.1), we make the transformation,

$$z = a \sinh u \sin\theta, \qquad r = a \cosh u \cos\theta, \qquad \phi = \phi, \qquad t = t, \tag{6.1}$$

a being a constant, we obtain quasi-oblate spheroidal coordinates. The metric (3.1) then becomes

$$\mathrm{d}s_+^2 = a^2\,\mathrm{e}^{2(\nu-\lambda)}(\sinh^2 u + \sin^2\theta)(\mathrm{d}u^2 + \mathrm{d}\theta^2) + a^2\,\mathrm{e}^{-2\lambda}\cosh^2 u\cos^2\theta\,\mathrm{d}\phi^2 - \mathrm{e}^{2\lambda}\,\mathrm{d}t^2. \tag{6.2}$$

The simplest solution to the Laplace equation for λ, in terms of these coordinates, is the monopole solution

$$\lambda = -\beta\tan^{-1}(\operatorname{cosech} u), \tag{6.3}$$

where β is a constant, which has been considered by Zipoy (1966) and by Bonnor and Sackfield (1968). Corresponding to this λ we have

$$\nu = \tfrac{1}{2}\beta^2\log\left(\frac{\sinh^2 u + \sin^2\theta}{\cosh^2 u}\right). \tag{6.4}$$

The ranges of the coordinates u, θ are $0 \leqslant u < \infty$, $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

Let us take as source of the field the oblate spheroidal shell Σ given by $u = u_0$, u_0 a positive constant. Since λ is constant on $u = u_0$ the interior metric is flat as in § 4 and, for the rest, one proceeds exactly as in that section. Corresponding to (4.14), the condition for Σ to be embeddable in the interior flat space–time is

$$\frac{(\sinh^2 u_0 + \sin^2\theta)^{(\beta^2+1)}}{(\cosh u_0)^{2\beta^2}} - \cosh^2 u_0 \sin^2\theta \geqslant 0, \tag{6.5}$$

for all θ, $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. This is satisfied if and only if

$$\sinh u_0 \geqslant \beta. \tag{6.6}$$

In other words, the situation here is qualitatively similar to case (b) of § 4. Case (c) of § 4, which allows a continuous sequence of equipotential shell sources over the full range of the parameter $x_0 (1 \leqslant x_0 < \infty)$, has no analogy in oblate spheroidal coordinates. Thus, the disc source of Bonnor and Sackfield is not a limiting case of the above since, by (6.6), u_0 is bounded away from zero. The mass of the shell, calculated by the integral (5.6) is, as before,

$$M = \beta a. \tag{6.7}$$

Hence, (6.6) represents an upper limit on the (mass/typical radius) ratio of the shell.

Another case of interest is when the exterior metric is that of Curzon, i.e. the metric (3.1) with

$$\lambda = -m(r^2 + z^2)^{-1/2}, \qquad \nu = -(m^2 r^2/2)(r^2 + z^2)^{-2}, \tag{6.8}$$

where m is a constant. If we take the shell Σ to be on the surface

$$r^2 + z^2 = a^2, \tag{6.9}$$

where a is a positive constant, λ is again constant on Σ and hence the interior metric is flat, as before. The condition for embeddability of Σ in the flat interior space–time is

$$\left(1-\frac{r^2}{a^2}\right)^{-1} e^{(-m^2r^2/a^4)} \geqslant 1 \tag{6.10}$$

for all r, $0 \leqslant r \leqslant a$. This is so if and only if,

$$m \leqslant a. \tag{6.11}$$

The mass of the shell, as calculated by (5.6), is m, so once again there is an upper limit to the (mass/typical radius) ratio of the shell, expressed by (6.11). Details for the Curzon case may be found in Hogan (1974).

7. Conclusion

In § 3, a general method has been given for constructing equipotential shell sources for given exterior static axially symmetric gravitational fields. This method has been applied to some particular exterior fields of interest and, for each of the fields treated, we have found a one-parameter sequence of equipotential sources. In cases (a) and (b) of § 4 and in both cases mentioned in § 6 there is, for a given mass, a lower limit to the radius of the source in any direction. In each of these cases, as the value of the relevant parameter decreases, the radius of the shell in all directions decreases. Case (c) of § 4 exhibits an exceptional type of behaviour, as shown, but, in this case, while the radius in the r direction goes to zero as the parameter x_0 tends to unity, the radius in the z direction goes to infinity. This behaviour may be indicative of a general result for more realistic sources where there is a volume distribution of matter.

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Appendix

In § 2, following Israel (1966), we have called $S_{\mu\nu}$ the 'surface energy tensor' of the shell. To see the relation of $S_{\mu\nu}$ to the (singular) energy tensor of the matter constituting the shell and how equation (2.5) comes about we may proceed as follows.

Introduce Gaussian normal coordinates $\{x^i\}$ based on Σ with $x^\mu = \xi^\mu$ ($\mu = 0, 2, 3$) and $x^1 = 0$ on Σ, while $x^1 = \pm$ (geodesic distance normal to Σ) for events in $V^\pm$, respectively. Let $\bar{g}_{\mu\nu}$ and $\bar{K}_{\mu\nu}$ be the intrinsic metric and extrinsic curvature tensors, respectively, on the subspaces $x^1 = \text{constant}$. Then

$$g_{\mu\nu} = \bar{g}_{\mu\nu}, \qquad \bar{K}_{\mu\nu} = \tfrac{1}{2}\bar{g}_{\mu\nu,1} \tag{A.1}$$

and

$$(g^\pm_{\mu\nu})_\Sigma = \lim_{x^1 \to 0\pm} \bar{g}_{\mu\nu}, \tag{A.2}$$

$$K^\pm_{\mu\nu} = \lim_{x^1 \to 0\pm} \bar{K}_{\mu\nu}. \tag{A.3}$$

The covariant components of the Einstein tensor in a neighbourhood of Σ are (cf, for example, Misner *et al* (1973, p 552), but note the differences in the sign conventions for the Ricci and extrinsic curvature tensors)

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + (\bar{K}_{\mu\nu} - \bar{g}_{\mu\nu}\bar{K})_{,1} + 3\bar{K}\bar{K}_{\mu\nu} - 2\bar{K}_{\mu\sigma}\bar{K}^{\sigma}_{\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}(\bar{K}^2 + \bar{K}_{\sigma\alpha}\bar{K}^{\sigma\alpha}), \tag{A.4}$$

$$G_{1\mu} = -\bar{K}^{\nu}_{\mu\|\nu} + \bar{K}_{,\mu}, \tag{A.5}$$

$$G_{11} = -\tfrac{1}{2}(\bar{R} + \bar{K}^2 - \bar{K}_{\mu\sigma}\bar{K}^{\mu\sigma}), \tag{A.6}$$

where a bar over a quantity indicates that it pertains to the subspaces x^1 = constant, $\bar{K} = \bar{K}^{\mu}_{\mu} = \bar{g}^{\mu\nu}\bar{K}_{\mu\nu}$ and the double stroke in (A.5) denotes a covariant derivative with respect to $\bar{g}_{\mu\nu}$.

The material energy tensor of the shell (see Misner *et al* 1973, Papapetrou and Hamoui 1968) will be of the form

$$T_{ij} = t_{ij}\,\delta(x^1), \tag{A.7}$$

where δ denotes the Dirac delta function. The surface energy tensor, S_{ij}, on Σ is defined by

$$S_{ij} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} T_{ij}\,\mathrm{d}x^1. \tag{A.8}$$

Einstein's field equations,

$$G_{ij} = -\kappa T_{ij},$$

together with (A.8) then yield

$$-\kappa S_{ij} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} G_{ij}\,\mathrm{d}x^1. \tag{A.9}$$

By (2.3) and (A.2), $\bar{g}_{\mu\nu}$ is continuous at Σ and hence $\bar{R}$ and $\bar{G}_{\mu\nu}$ are continuous at Σ. Furthermore, the second of (A.1) implies that $\bar{K}_{\mu\nu}$ contains no delta functions at Σ. Thus, (A.9) together with (A.3)–(A.6) yield

$$-\kappa S_{11} = 0, \tag{A.10}$$

$$-\kappa S_{1\mu} = 0, \tag{A.11}$$

$$-\kappa S_{\mu\nu} = \gamma_{\mu\nu} - g_{\mu\nu}\gamma^{\sigma}_{\sigma}, \tag{A.12}$$

where $\gamma_{\mu\nu}$ is as in (2.4) and $g_{\mu\nu} = \bar{g}_{\mu\nu}(x^1 = 0)$, i.e. in (A.12) $g_{\mu\nu}$ has the meaning attached to it in § 2.

The surface energy tensor defined by (A.8) has therefore no components normal to Σ and is essentially a tensor field on Σ with components $S_{\mu\nu}$ given by (A.12). In § 2 we have called this tensor field 'the surface energy tensor of the shell'.

From the above it is clear that one may adapt Whittaker's (1935) theorem to obtain equation (5.6) for the total mass of the shell.

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